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DGG-003-016401

Seat No. ________

M. Sc. (Sem. IV) (Mathematics) (CBCS) Examination

April / May – 2015

MATHS - CMT - 4001 : Commutative Ring Theory

(Old Course)

Faculty Code : 003

Subject Code : 016401

Time : $2\frac{1}{2}$ Hours] [Total Marks : **70**

Instructions : (1) Answer all the questions.

(2) Each question carries 14 marks.

1. Answer any **Seven** $7 \times 2 = 14$

(a) Define the Jacobson radical $J(R)$ of a ring R. Verify that $J(F[X]) = (0)$, where $F[X]$ is the polynomial ring in one variable X over a field F.

(b) Define a unit in a ring R. Determine the units of $\mathbf{Z}$.

(c) When is a module M over a ring R said to be finitely generated? Illustrate it with an example.

(d) Define a multiplicatively closed subset of a ring. If P is any prime ideal of a ring R, then verify that $R \backslash P$ is a multiplicatively closed subset of R.

(e) State Nakayama's lemma.

(f) Define primary ideal of a ring. If p is a prime number, then show that $p^n\mathbf{Z}$ is a primary ideal of $\mathbf{Z}$ for any $n \geq 1$.

(g) Show that the real number $\frac{1+\sqrt{37}}{2}$ is integral over $\mathbf{Z}$.

(h) Define a Noetherian module.

(i) Define a ring homomorphism. Verify that any ring homomorphism from a field F into a nonzero ring is injective.

(j) When is a property of a ring said to be a local property?

2. Answer any **Two** $2 \times 7 = 14$

(a) Prove that the set of all nilpotent elements in a ring R is an ideal of R. If $nil(R)$ is the nilradical of R, then prove that the ring $\frac{R}{nil(R)}$ has no nonzero nilpotent element.

(b) (i) Let $P_1, \ldots, P_n$ be prime ideals of a ring R. If an ideal I of R is such that $I \subseteq \cup_{i=1}^{n} P_i$, then show that $I \subseteq P_i$ for some $i \in \{1, \ldots, n\}$.

(ii) Let $I_1, \ldots, I_n$ be ideals of a ring R. If a prime ideal P of R satisfies $P \supseteq \cap_{i=1}^{n} I_i$, then prove that $P \supseteq I_i$ for some $i \in \{1, \ldots, n\}$.

(c) Let M be a finitely generated R-module and let I be an ideal of R. If f is an R-module endomorphism of M such that $f(M) \subseteq IM$, then prove that f satisfies an equation of the form $f^n + a_1 f^{n-1} + \cdots + a_n Id_M =$ zero

R-homomorphism from M into M for some $n \geq 1$, where $a_i \in I$ for each $i \in \{1, \ldots, n\}$, and $Id_M : M \to M$ is the identity map.

3. (*a*) Let M be an R-moule. Prove that $M = (0)$ if and only if $M_P = (0)$ for every prime ideal P of R. 5

(*b*) Let M_1, M_2 be submodules of an R-module M. Prove that $\frac{M_1+M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$ as R-modules. 5

(*c*) Let P be any nonzero prime ideal of a principal ideal domain R. Prove that P is a maximal ideal of R. 4

OR

3. (*a*) Let S be a multiplicatively closed subset of a ring R. For any ideal I of R, prove that $S^{-1}(\sqrt{I}) = \sqrt{S^{-1}I}$. 5

(*b*) Let I be a decomposable ideal of a ring R. Let $I = \cap_{i=1}^{n} q_i$ be a minimal primary decomposition of I with $\sqrt{q_i} = P_i$ for each $i \in \{1, \ldots, n\}$. Prove that $\cup_{i=1}^{n} P_i = \{r \in R | (I :_R r) \neq I\}$. 5

(*c*) Let R be a subring of a ring T. If $t \in T$ is integral over R, then prove that $R[t]$ is a finitely generated R-module. 4

4. Answer any **Two** $2 \times 7 = 14$

(*a*) State and prove the first uniqueness theorem on decomposable ideals in a ring.

(*b*) Let M be a module over a ring R. If a submodule N of M is such that N and $\frac{M}{N}$ are Artinian, then prove that M is Artinian.

(*c*) Let R be a Noetherian ring. If I is a proper irreducible ideal of R, then prove that I is primary.

5. Answer any **Two** $2 \times 7 = 14$

(*a*) State and prove the Chinese remainder theorem.

(*b*) Let S be a multiplicatively closed subset of a ring R. Let $g : R \to T$ be a ring homomorphism such that $g(s)$ is a unit in T for all $s \in S$. Prove that there exists a unique ring homomorphism $h : S^{-1}R \to T$ such that $h(\frac{r}{1}) = g(r)$ for all $r \in R$.

(*c*) Let R be a subring of a ring T. Let S be a multiplicatively closed subset of R. If C is the integral closure of R in T, then prove that $S^{-1}C$ is the integral closure of $S^{-1}R$ in $S^{-1}T$.

(*d*) Prove that the nilradical of an Artin ring R, is nilpotent.